AN ERSATZ EXISTENCE THEOREM FOR FULLY NONLINEAR PARABOLIC EQUATIONS WITHOUT CONVEXITY ASSUMPTIONS

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ABSTRACT. We show that for any uniformly parabolic fully nonlinear second-order equation with bounded measurable "coefficients" and bounded "free" term in the whole space or in any cylindrical smooth domain with smooth boundary data one can find an approximating equation which has a continuous solution with the first and the second spatial derivatives under control: bounded in the case of the whole space and locally bounded in case of equations in cylinders. The approximating equation is constructed in such a way that it modifies the original one only for large values of the second spatial derivatives of the unknown function. This is different from a previous work of Hongjie Dong and the author where the modification was done for large values of the unknown function and its spatial derivatives.

1. INTRODUCTION

This article is a natural continuation of [2] and is written in the same framework. We are given a function $H(u, t, x)$,

$$(u = (u', u''), \quad u' = (u'_0, u'_1, ..., u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathbb{S}, \quad (t, x) \in \mathbb{R}^{d+1},$$

where $\mathbb{S}$ is the set of symmetric $d \times d$ matrices, and we are dealing with some modifications of the parabolic equation

$$\partial_t v(t, x) + H[v](t, x) := \partial_t v(t, x) + H(v(t, x), Dv(t, x), D^2 v(t, x), t, x) = 0 \tag{1.1}$$

in subdomains of $(0, T) \times \mathbb{R}^d$, where $T \in (0, \infty)$,

$$\mathbb{R}^d = \{x = (x_1, ..., x_d) : x_1, ..., x_d \in \mathbb{R}\},$$

$$\partial_t = \frac{\partial}{\partial t}, \quad D^2 u = (D_{ij} u), \quad Du = (D_i u), \quad D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_i D_j.$$

As in [2] we are looking for a uniformly elliptic operator $P[v]$ given by a convex positive-homogeneous of degree one function P independent of (t, x)

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such that the boundary-value problem we are interested in for the equation

$$\partial_t v + \max(H[v], P[v] - K) = 0 \qquad (1.2)$$

would be solvable in classical sense (a.e.) for any constant $K \geq 0$. However, unlike [2] we do not allow $P[v]$ to depend on v and its first derivatives, so that $P(u, t, x) = P(u'')$. A big advantage of this approach is that we do not need Lipschitz continuity of $H(u, t, x)$ with respect to u' but rather not faster than linear growth of $H(u', 0, t, x)$ as $|u'| \to \infty$. Actually, our results even in the particular case of H *independent* of u' will play a major role in a subsequent article aimed at proving that L_p-viscosity solutions of (1.1) are in $C^{1+\alpha}$ provided that "the main coefficients" of H are in VMO.

Solvability theory for uniformly nondegenerate parabolic equations like (1.1) and its elliptic counterparts in Hölder classes of functions is well developed in case H is convex or concave in u'' (see, for instance, [4], [6], [12]). In case this condition is abandoned N. Nadirashvili and S. Vlăduţ [13] gave an example of elliptic fully nonlinear equation which does not admit classical (or even $C^{1+\alpha}$ viscosity) solution. For that reason the interest in Sobolev space theory became even more justifiable. In [10] the author proved the first existence (and uniqueness) result for fully nonlinear elliptic equations under relaxed convexity assumption for equations with VMO "coefficients". Previously, M. G. Crandall, M. Kocan, and A. Święch [1] established the *solvability* in local Sobolev spaces of the boundary-value problems for fully nonlinear parabolic equations and N. Winter [14] established the solvability in the global W_p^2-space of the associated boundary-value problem in the elliptic case. In the solvability parts of these two papers H is assumed to be convex in u'' and, basically, have continuous "coefficients" (actually, it is assumed to be uniformly sufficiently close to the ones having continuous "coefficients").

There is also a quite extensive a priori estimates side of the story (not involving the solvability) for which we refer the reader to [1], [2], [14] and the references therein.

Apart from Theorems 2.1 about the solvability of equations in the whole space and 2.2 about that in cylinders, which are proved in Sections 4 and 5, respectively, Theorem 2.3 proved in Section 6 is also one of our main results. Roughly speaking, it says that as $K \to \infty$ the solutions of (1.2) converge to the maximal L_{d+1}-viscosity solution of (1.1). The existence of the maximal L_p-viscosity solution for elliptic case was proved in [5]. We provide a method which in principle allows one to find it. Finally, Section 2 contains our main results and Section 3 is devoted to reducing Theorem 2.1 to a simpler statement.

2. Main results

Fix some constants $\delta \in (0, 1)$ and $K_0 \in [0, \infty)$. Set

$$\mathbb{S}_\delta = \{a \in \mathbb{S} : \delta|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d\},$$

where and everywhere in the article the summation convention is enforced.

Assumption 2.1. (i) The function $H(u, t, x)$ is measurable with respect to (u', t, x) for any u'', Lipschitz continuous in u'', and at all points of differentiability of H with respect to u'' we have $H_{u''} \in \mathbb{S}_\delta$,

(ii) The number

$$\bar{H} := \sup_{u', t, x} \left(|H(u', 0, t, x)| - K_0 |u'| \right) \quad (\geq 0)$$

is finite,

(iii) There is an increasing continuous function $\omega(r)$, $r \geq 0$, such that $\omega(0) = 0$ and

$$|H(u', u'', t, x) - H(v', u'', t, x)| \leq \omega(|u' - v'|)$$

for all u, v, t, and x.

By Theorem 3.1 of [8] there exists a set

$$\Lambda = \{l_1, ..., l_m\} \subset \mathbb{Z}^d,$$

$m = m(\delta, d) \geq d$, chosen on the sole basis of knowing δ and d and there exists a constant

$$\hat{\delta} = \hat{\delta}(\delta, d) \in (0, \delta/4]$$

such that:

(a) We have $l_i = e_i$ and

$$e_i \pm e_j \in \{l_1, ..., l_m\} = \{-l_1, ..., -l_m\}$$

for all $i, j = 1, ..., d$, where $e_1, ..., e_d$ is the standard orthonormal basis of $\mathbb{R}^d$;

(b) There exist real-analytic functions $\lambda_1(a), ..., \lambda_m(a)$ on $\mathbb{S}_{\delta/4}$ such that for any $a \in \mathbb{S}_{\delta/4}$

$$a = \sum_{k=1}^m \lambda_k(a) l_k l_k^*, \quad \hat{\delta}^{-1} \geq \lambda_k(a) \geq \hat{\delta}, \quad \forall k. \tag{2.1}$$

Introduce

$$\mathcal{P}(z'') = \max_{\substack{\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ k=1,...,m}} \sum_{k=1}^m a_k z_k'',$$

and for $u'' \in \mathbb{S}$ define

$$P(u'') = \mathcal{P}(\langle u'' l_1, l_1 \rangle, ..., \langle u'' l_m, l_m \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. Naturally, by $P[v]$ we mean a differential operator constructed as in (1.1).

Let Ω be an open bounded subset of $\mathbb{R}^d$ with C^2 boundary. We denote the parabolic boundary of the cylinder $\Omega_T = (0, T) \times \Omega$ by

$$\partial' \Omega_T = (\partial \Omega_T) \setminus (\{0\} \times \Omega).$$

Below, for $\alpha \in (0, 1)$, the parabolic spaces $C^{\alpha/2, \alpha}$ and elliptic spaces C^α are usual Hölder spaces. These spaces are provided with natural norms.

Theorem 2.1. *Let $K \geq 0$ be a fixed constant, $g \in W^{1,2}_\infty(\Omega_T) \cap C(\bar{\Omega}_T)$. Suppose that Assumption 2.1 is satisfied. Then equation (1.2) in Ω_T with boundary condition $v = g$ on $\partial'\Omega_T$ has a solution $v \in C(\bar{\Omega}_T) \cap W^{1,2}_{\infty,loc}(\Omega_T)$. In addition, for all i, j, and $p \in (d+1, \infty)$,*

$$|v|, |D_i v|, \rho |D_{ij} v|, |\partial_t v| \leq N(\bar{H} + K + \|g\|_{W^{1,2}_\infty(\Omega_T)}) \quad in \quad \Omega_T \quad (a.e.), \quad (2.2)$$

$$\|v\|_{W^{1,2}_p(\Omega_T)} \leq N_p(\bar{H} + K + \|g\|_{W^{1,2}_p(\Omega_T)}), \quad (2.3)$$

$$\|v\|_{C^{\alpha/2,\alpha}(\Omega_T)} \leq N(\bar{H} + \|g\|_{C^{\alpha/2,\alpha}(\Omega_T)}), \quad (2.4)$$

where

$$\rho = \rho(x) = \text{dist}\,(x, \mathbb{R}^d \setminus \Omega),$$

$\alpha \in (0,1)$ is a constant depending only on d and δ, N is a constant depending only on Ω, T, K_0, and δ, whereas N_p only depends on Ω, T, K_0, δ, and p (in particular, N and N_p are independent of ω).

Here is our second main result.

Theorem 2.2. *Let $K \geq 0$ be a fixed constant, and $g \in W^2_\infty(\mathbb{R}^d)$. Then the equation (1.2) in $Q_T := [0, T] \times \mathbb{R}^d$ (a.e.) with terminal condition $v(T, x) = g(x)$ has a solution $v \in W^{1,2}_\infty(Q_T) \cap C(Q_T)$. In addition,*

$$|v|, |Dv|, |D^2 v|, |\partial_t v| \leq N(\bar{H} + K + \|g\|_{W^2_\infty(\mathbb{R}^d)}) \quad in \quad Q_T \quad (a.e.),$$

$$\|v\|_{C^{\alpha/2,\alpha}(Q_T)} \leq N(\bar{H} + \|g\|_{C^\alpha(\mathbb{R}^d)}),$$

where $\alpha \in (0,1)$ is a constant depending only on d and δ and N is a constant depending only on T, K_0, d, and δ.

Before stating our third main result introduce the following.

Assumption 2.2. The function H is a nonincreasing function of u'_0, which is continuous with respect to u'_0 uniformly with respect to other variables, and is Lipschitz continuous with respect to $(u'_1, ..., u'_d)$ with constant independent of other variables.

Observe that under Assumption 2.2 the solutions $v = v_K$ constructed in Theorem 2.1 for each K are unique and decrease as $K \to \infty$.

Theorem 2.3. *Suppose that Assumptions 2.1 and 2.2 are satisfied. Then, as $K \to \infty$, v_K converges uniformly on $\bar{\Omega}_T$ to a continuous function v which, in terminology of [1], is an L_{d+1}-viscosity solutions of (1.2) in Ω_T with boundary condition $v = g$ on $\partial'\Omega_T$. Furthermore, v is the maximal L_{d+1}-viscosity subsolution of class $C(\bar{\Omega}_T)$ of this problem.*

3. Reduction of Theorem 2.1 to a simpler case

Denote by $C^{1,2}(\bar{\Omega}_T)$ the set of functions $g(t,x)$ such that $g, Dg, D^2g, \partial_t g \in C(\bar{\Omega}_T)$. The norm in $C^{1,2}(\bar{\Omega}_T)$ is introduced in an obvious way.

Lemma 3.1. *Suppose that the assertions of Theorem 2.1 hold true if $g \in C^{1,2}(\bar{\Omega}_T)$ and, in addition to Assumption 2.1, for any $s, t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, $u = (u', u'')$, and $v = (v', v'')$,*

$$|H(u,t,x) - H(u,s,y)| \leq N'(|t-s| + |x-y|)(1+|u|), \qquad (3.1)$$

$$|H(u', u'', t, x) - H(v', u'', t, x)| \leq N'|u' - v'| \qquad (3.2)$$

where N' is independent of $t, s, x, y, u,$ and v. Then the assertions of Theorem 2.1 hold true without these additional assumptions as well.

Proof. First we assume that the assertions of Theorem 2.1 hold true with g as there but under the additional assumption that (3.1) and (3.2) hold.

Note that

$$|H(u,t,x)| \leq |H(u,t,x) - H(u',0,t,x)| + K_0|u'| + \bar{H} \leq \bar{H} + N(K_0, d, \delta)|u|. \tag{3.3}$$

Then let B_1 be the open unit ball in $\mathbb{R}^{d+1}$ centered at the origin. Take a nonnegative $\zeta \in C_0^\infty(B_1)$, which integrates to one and introduce $H_n(u,t,x)$ as the convolution of $H(u,t,x)$ and $n^{d+1}\zeta(nt, nx)$ performed with respect to (t,x). Observe that H_n satisfies Assumption 2.1 with the same constant δ, whereas

$$|H_n(u,t,x) - H_n(u,s,y)| \leq n|B_1|(|t-s| + |x-y|)\sup_z |H(u,z)| \sup|D\zeta|,$$

where $|B_1|$ is the volume of B_1, and (3.1) is satisfied due to (3.3).

Next, define $H^n(u,t,x)$ as the convolution of $H_n(u,x)$ and $n^{d+1}\zeta(nu', t, x)$ performed with respect to u'. Obviously, for each n, H^n satisfies (3.1) with a constant N'. Furthermore

$$H^n_{u'_k}(u', u'', t, x) = n \int_{\mathbb{R}^{d+1}} H_n(u' - v'/n, u'', t, x)\zeta_{u'_k}(v')\, dv'$$

$$= n \int_{\mathbb{R}^{d+1}} [H_n(u' - v'/n, u'', t, x) - H_n(u', u'', t, x)]\zeta_{u'_k}(v')\, dv'.$$

It follows that

$$|H^n_{u'_k}(u,t,x)| \leq n\omega(1/n)|B_1|\sup|D\zeta|,$$

so that H^n also satisfies (3.2).

Now by assumption there exist solutions $v^n \in C(\bar{\Omega}_T) \cap W^{1,2}_{\infty,\text{loc}}(\Omega_T)$ of

$$\partial_t v^n + \max(H^n[v^n], P[v^n] - K) = 0 \tag{3.4}$$

in Ω_T (a.e.) with boundary condition $v^n = g$, for which estimates (2.2), (2.3), and (2.4) hold with v^n in place of v with the constants N and N_p from Theorem 2.1 and with

$$\bar{H}^n = \sup_{u',t,x} \left(|H^n(u',0,t,x)| - K_0|u'|\right) \qquad (\leq \bar{H} + K_0 n^{-1})$$

in place of $\bar{H}$. Furthermore, being uniformly bounded and uniformly continuous, the sequence $\{v^n\}$ has a subsequence uniformly converging to a function v, for which (2.2), (2.3), and (2.4), of course, hold and $v \in C(\bar{\Omega}_T) \cap W_{\infty,\mathrm{loc}}^{1,2}(\Omega_T)$. For simplicity of notation we suppose that the whole sequence v^n converges.

Observe that

$$\partial_t v^m + \check{H}_K^n[v^m] \geq 0 \tag{3.5}$$

in Ω_T (a.e.) for all $m \geq n$, where

$$\check{H}_K^n(u, t, x) := \sup_{k \geq n} \max(H^k(v^k(t,x), Dv^k(t,x), u'', t, x), P(u'') - K).$$

In light of (3.5) and the fact that the norms $\|v^n\|_{W_p^{1,2}(\Omega_T)}$ are bounded, by Theorem 3.5.9 of [6] we have

$$\partial_t v + \check{H}_K^n[v] \geq 0 \tag{3.6}$$

in Ω_T (a.e.).

Now we notice that by embedding theorems Dv^k are locally uniformly continuous in Ω_T and this and the convergence $v^n \to v$ implies by a standard fact of calculus that Dv^k converge to Dv locally uniformly in Ω_T. Also

$$|H^k(u, t, x) - H_k(u, t, x)| \leq \omega(1/k),$$

$$|H_k(v^k(t,x), Dv^k(t,x), D^2v(t,x), t, x) - H_k(v(t,x), Dv(t,x), D^2v(t,x), t, x)|$$
$$\leq \omega(|v^k - v|(t,x) + |Dv^k - Dv|(t,x)),$$

which along with what was said above implies that

$$\partial_t v + \hat{H}_K^n[v] \geq -\varepsilon_n \tag{3.7}$$

in Ω_T (a.e.), where the functions $\varepsilon_n \to 0$ in Ω_T (even locally uniformly) and

$$\hat{H}_K^n(u, t, x) := \sup_{k \geq n} \max(H_k(u, t, x), P(u'') - K).$$

Then we notice that by the Lebesgue differentiation theorem for any u

$$\lim_{n \to \infty} \hat{H}_K^n(u, t, x) = \max(H(u, t, x), P(u) - K) \tag{3.8}$$

for almost all (t, x). Since for any bounded set Γ in the range of u, $\hat{H}_K^n(u, t, x)$ are uniformly continuous on Γ uniformly with respect to (t, x) and n, there exists a subset of Ω_T of full measure such that (3.8) holds on this subset for all u.

We conclude that in Ω_T (a.e.)

$$\partial_t v + \max(H[v], P[v] - K) \geq 0. \tag{3.9}$$

The opposite inequality is obtained by considering

$$\inf_{k \geq n} \max(H^k(v^k(t,x), Dv^k(t,x), u'', t, x), P(u'') - K).$$

The fact that it suffices to prove Theorem 2.1 under the additional assumption that $g \in C^{1,2}(\bar{\Omega}_T)$ is proved by mollifying g and using a very simplified version of the above arguments. The lemma is proved.

Next, we show that one may assume that H is boundedly inhomogeneous with respect to u''. Introduce

$$P_0(u) = P_0(u'') = \max_{a \in \mathbb{S}_{\delta/2}} a_{ij} u''_{ij},$$

where the summation is performed before the maximum is taken. It is easy to see that $P_0[u]$ is Pucci's operator:

$$P_0(u) = -(\delta/2) \sum_{k=1}^{d} \lambda_k^-(u'') + 2\delta^{-1} \sum_{k=1}^{d} \lambda_k^+(u''),$$

where $\lambda_1(u''), ..., \lambda_d(u'')$ are the eigenvalues of u'' and $a^{\pm} = (1/2)(|a| \pm a)$.

Observe that

$$P(u) = \max_{\substack{\hat{\delta}/2 \le a_k \le 2\hat{\delta}^{-1} \\ k=1,...,m}} \sum_{i,j=1}^{d} \sum_{k=1}^{m} a_k l_{ki} l_{kj} u''_{ij}.$$

Moreover, owing to property (b) in Section 2, the collection of matrices

$$\sum_{k=1}^{m} a_k l_k l_k^*$$

such that $\hat{\delta} \le a_k \le \hat{\delta}^{-1}, k = 1, ..., m$, covers $\mathbb{S}_{\delta/4}$. Hence,

$$P(u) \ge -(\delta/4) \sum_{k=1}^{d} \lambda_k^-(u'') + 4\delta^{-1} \sum_{k=1}^{d} \lambda_k^+(u'')$$

$$\ge P_0(u) + (\delta/4) \sum_{k=1}^{d} |\lambda_k(u'')|. \tag{3.10}$$

In particular, $P_0 \le P$ and therefore,

$$\max(H, P - K) = \max(H_K, P - K),$$

where $H_K = \max(H, P_0 - K)$. It is easy to see that the function H_K satisfies Assumption 2.1 (i) with $\delta/2$ in place of δ, satisfies Assumption 2.1 (iii) with the same function ω, and

$$|H_K(u', 0, t, x)| \le |H(u', 0, t, x)| \le K_0|u'| + \bar{H},$$

so that the number

$$\bar{H}_K := \sup_{u', t, x} \left(|H_K(u', 0, t, x)| - K_0|u'| \right) \quad (\le \bar{H})$$

is finite and Assumption 2.1 (ii) is also satisfied. Also observe that H_K satisfies (3.1) and (3.2) with the same constant N'.

To continue we note the following.

Lemma 3.2. *There is a constant $\kappa > 0$ depending only on δ and d such that*

$$H \le P_0 - \kappa|u''| + K_0|u'| + \bar{H}_{(+)}, \tag{3.11}$$

$$H_K \le P - \kappa|u''| + K_0|u'| + \bar{H}_{(+)}, \tag{3.12}$$

where

$$\bar{H}_{(+)} := \sup_{u',t,x} \left(H^+(u',0,t,x) - K_0|u'| \right) \le \bar{H}_K.$$

Furthermore, H_K is boundedly inhomogeneous with respect to u'' in the sense that at all points of differentiability of $H_K(u,t,x)$ with respect to u''

$$|H_K(u,t,x) - H_{Ku''_{ij}}(u,t,x)u''_{ij}| \le N(K_0+1)(\bar{H}_K + K + |u'|), \tag{3.13}$$

where N depends only on d and δ.

Proof. One proves (3.11) by writing

$$H(u,t,x) = [H(u,t,x) - H(u',0,t,x)] + H(u',0,t,x),$$

transforming the first term on the right by using Hadamard's formula (cf. the proof of Lemma 3.1 in [9]) and observing that

$$H(u',0,t,x) \le (H^+(u',0,t,x) - K_0|u'|) + K_0|u'|.$$

Estimate (3.12) now also follows since $P_0 \le P$. To prove (3.13) note that if

$$\kappa|u''| \ge \bar{H}_{(+)} + K + K_0|u'|, \tag{3.14}$$

then by (3.11)

$$H(u,x) \le P_0(u) - \kappa|u''| + K_0|u'| + \bar{H}_{(+)} \le P_0(u) - K,$$

so that $H_K(u,t,x) = P_0(u) - K$ and the left-hand side of (3.13) is just K owing to the fact that P_0 is positive homogeneous of degree one. On the other hand, if the opposite inequality holds in (3.14), then it follows from

$$|H_K(u,t,x)| \le |H_K(u,t,x) - H_K(u',0,t,x)| + |H_K(u',0,t,x)|$$

$$\le N|u''| + \bar{H}_K + K_0|u'| \le N(K_0+1)(|u'| + K + \bar{H}_K + \bar{H}_{(+)})$$

that the left-hand side of (3.13) is dominated by

$$N(K_0+1)(|u'| + K + \bar{H}_K + \bar{H}_{(+)}).$$

After that it only remains to notice that

$$H(u',0,t,x) \le \max(H(u',0,t,x), -K) = H_K(u',0,t,x),$$

$$H^+(u',0,t,x) \le |H_K(u',0,t,x)|, \quad \bar{H}_{(+)} \le \bar{H}_K.$$

The lemma is proved.

This lemma shows that in the rest of the proof of Theorem 2.1 we may assume that not only Assumption 2.1 is satisfied with $\delta/2$ in place of δ and (3.1) and (3.2) hold with a constant N', but also at all points of differentiability of H with respect to u

$$|H(u,t,x) - H_{u''_{ij}}(u,t,x)u''_{ij}| \le K'_0(\bar{H} + K + |u'|), \tag{3.15}$$

$$|H(u,t,x) - H_{u''_{ij}}(u,t,x)u''_{ij}| \le K_1(1+|u'|), \qquad (3.16)$$

where $K'_0 = N(\delta,d)(K_0+1)$ and $K_1 = K'_0(\bar{H}+K+1)$, and

$$H \le P - \kappa|u''| + K_0|u'| + \bar{H}, \qquad (3.17)$$

where κ is the constant from Lemma 3.2.

As a result of the above arguments we see that to prove Theorem 2.1 it suffices to prove the following.

Theorem 3.3. *Suppose that $g \in C^{1,2}(\bar{\Omega}_T)$ and Assumption 2.1 is satisfied with $\delta/2$ in place of δ. Also assume that* (3.15) *(and hence* (3.16)*) holds at all points of differentiability of $H(u,t,x)$ with respect to u. Finally, assume that estimates* (3.1) *and* (3.2) *with a constant N' and* (3.17) *hold for any $t,s \in \mathbb{R}$, $x,y \in \mathbb{R}^d$, and u,v. Then the assertions of Theorem 2.1 hold true.*

4. PROOF OF THEOREM 3.3

By Theorem 5.1 and Corollary 5.2 of [8], where we take

$$I = [-K_1, K_1], \quad J = [-2K_1, 2K_1], \quad C'' = I \times \mathbb{S}_{\delta/2}, \quad B'' = J \times \mathbb{S}_{\delta/4},$$

there exists a measurable function $\mathcal{H}(u', z'', t, x)$, $u' \in \mathbb{R}^{d+1}$, $z'' \in \mathbb{R}^m$ (recall (2.1)), $(t,x) \in \mathbb{R}^{d+1}$, such that it is Lipschitz continuous with respect to z'' with constant independent u' and of (t,x),

$$H(u,t,x) = \mathcal{H}(u', \langle u''l_1, l_1 \rangle, ..., \langle u''l_m, l_m \rangle, t, x) \qquad (4.1)$$

for all values of arguments, and at all points of differentiability of $\mathcal{H}$ with respect to z'' we have

$$D_{z''}\mathcal{H}(u', z'', t, x) \in [\hat{\delta}, \hat{\delta}^{-1}]^m, \qquad (4.2)$$

$$(1+|u'|)^{-1}[\mathcal{H}(u', z'', t, x) - \langle z'', D_{z''}\mathcal{H}(u', z'', t, x) \rangle] \in J, \qquad (4.3)$$

$$|\mathcal{H}(u', z'', t, x) - \mathcal{H}(u', z'', s, y)| \le N(|t-s| + |x-y|)(1+|z''|+|u'|), \quad (4.4)$$

where N is a constant independent of u', z'', t, x, s, y.

We are going to use finite-difference approximations of the operators $H[v]$ and $P[v]$ and for $h > 0$ and vectors l introduce

$$T_{h,l}\phi(x) = \phi(x+hl), \quad \delta_{h,l} = h^{-1}(T_{h,l}-1), \quad \Delta_{h,l} = h^{-2}(T_{h,l}-2+T_{h,-l}).$$

Also set

$$P_h[v](t,x) = \mathcal{P}(\delta_h^2 v(t,x)),$$

where

$$\delta_h^2 v = (\Delta_{h,l_1}v, ..., \Delta_{h,l_m}v).$$

Similarly we introduce

$$H_h[v](t,x) = \mathcal{H}(v(t,x), \delta_h v(t,x), \delta_h^2 v(t,x)),$$

where

$$\delta_h v = (\delta_{h,e_1}v, ..., \delta_{h,e_d}v),$$

and $H_{K,h}[v] = \max(H_h[v], P_h[v] - K)$.

Owing to (4.2) and Assumption 2.1 (ii) we have the following.

Lemma 4.1. *For all values of arguments*

$$\mathcal{H} \leq \mathcal{P} - (\hat{\delta}/2) \sum_{k=1}^{m} |z_k''| + K_0 |u'| + \bar{H}.$$

Introduce B as the smallest closed ball containing Λ and set

$$\Omega^h = \{x \in \Omega : x + hB \subset \Omega\} = \{x : \rho(x) > \lambda h\},$$

where λ is the radius of B.

Consider the equation

$$\partial_t v + H_{K,h}[v] = 0 \quad \text{in} \quad [0,T] \times \Omega^h \tag{4.5}$$

with boundary condition

$$v = g \quad \text{on} \quad \Big(\{T\} \times \Omega^h\Big) \cup \Big([0,T] \times (\bar{\Omega} \setminus \Omega^h)\Big). \tag{4.6}$$

In view of Picard's method of successive iterations, for any $h > 0$ there exists a unique bounded solution $v = v_h$ of (4.5)–(4.6). Furthermore, $\partial_t v_h(t,x)$ is bounded and is continuous with respect to t for any x.

Below in this section by h_0 and N with occasional indices we denote various (finite positive) constants depending only on Ω, $\{l_1, ..., l_m\}$, d, K_0, T, and δ, unless specifically stated otherwise.

Denote

$$\Lambda_1 = \Lambda, \quad \Lambda_{n+1} = \Lambda_n + \Lambda, \quad n \geq 1, \quad \Lambda_\infty = \bigcup_n \Lambda_n, \quad \Lambda_\infty^h = h\Lambda_\infty.$$

Observe that the set of points in Λ_∞^h lying in any bounded domain is finite since the l_i's have integral coordinates.

We need a particular case of Theorem 4.3 of [2]. Let Q^o be a nonempty subset of $(0,T) \times \Lambda_\infty^h$, which is open in the relative topology of $(0,T) \times \Lambda_\infty^h$. We introduce $\hat{Q}^o$ as the set of points $(t_0, x_0) \in (0,T] \times \Lambda_\infty^h$ for each of which there exists a sequence $t_n \uparrow t_0$ such that $(t_n, x_0) \in Q^o$. Observe that $Q^o \subset \hat{Q}^o$. Also define

$$Q = \hat{Q}^o \cup \{(t, x + h\Lambda) : (t,x) \in Q^o\}.$$

For $x \in \Lambda_\infty^h$ we denote by $Q^o_{|x}$ the x-section of Q^o: $\{t : (t,x) \in Q^o\}$. Assume that

$$Q^o \subset G := \{(t,x) \in \Omega_T^h : (\hat{\delta}/2) \sum_{k=1}^{m} |\Delta_{h,l_k} v_h(t,x)| >$$

$$> \bar{H} + K + K_0\big(|v_h(t,x)| + M_h(t,x)\big)\}, \tag{4.7}$$

where

$$M_h(t,x) = \sum_{k=1}^{m} |\delta_{h,l_k} v_h(t,x)|,$$

so that, owing to Lemma 4.1, $\partial_t v_h + P_h[v_h] - K \leq 0$ in $[0,T] \times \Omega^h$ and

$$\partial_t v_h + P_h[v_h] = K \quad \text{in} \quad Q^o. \tag{4.8}$$

To proceed with estimating $\partial_t v_h$ and second-order differences of v_h we introduce the following. Take a function $\eta \in C^\infty(\mathbb{R}^d)$ with bounded derivatives, such that $|\eta| \leq 1$ and set $\zeta = \eta^2$,

$$|\eta'(x)|_h = \sup_k |\delta_{h,l_k}\eta(x)|, \quad |\eta''(x)|_h = \sup_k |\Delta_{h,l_k}\eta(x)|,$$

$$\|\eta'\|_h = \sup_{\Lambda_\infty^h} |\eta'|_h, \quad \|\eta''\|_h = \sup_{\Lambda_\infty^h} |\eta''|_h.$$

Here is a particular case of Theorem 4.3 of [2] we need.

Lemma 4.2. *Assume that $Q \subset [0,T] \times \Omega^h$. Then there exists a constant $N = N(m,\delta) \geq 1$ such that on Q^o for any $k = 1,...,m$*

$$\zeta^2[(\Delta_{h,l_k}v_h)^-]^2 \leq \sup_{Q\backslash Q^o} \zeta^2[(\Delta_{h,l_k}v_h)^-]^2 + N(\|\eta''\|_h + \|\eta'\|_h^2)\bar{W}_k,$$

where

$$\bar{W}_k = \sup_Q(|\delta_{h,l_k}v_h|^2 + |\delta_{h,-l_k}v_h|^2).$$

Lemma 4.3. *There are constants $h_0 > 0$ and N such that for all $h \in (0, h_0]$*

$$|v_h - g| \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)})\rho, \tag{4.9}$$

$$|\partial_t v_h| \leq N(\bar{M}_h + \bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}) \tag{4.10}$$

on $\bar{\Omega}_T$, where $\bar{M}_h := \sup_{[0,T]\times\Omega^h} M_h$.

Proof. To prove (4.9) observe that by Hadamard's formula

$$0 = \partial_t v_h + H_{K,h}[v_h] = \partial_t v_h$$

$$+ \max\left(\mathcal{H}(v_h, \delta_h v_h, \delta_h^2 v_h, t, x), P_h[v_h] - K\right) - \max\left(\mathcal{H}(v_h, \delta_h v_h, 0, t, x), -K\right)$$

$$+ \max\left(H(v_h, \delta_h v_h, 0, t, x), -K\right)$$

$$= \partial_t v_h + \sum_{k=1}^m a_k \Delta_{h,l_k}v_h + f(v_h, \delta_h v_h, t, x), \tag{4.11}$$

where a_k are some functions satisfying $\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1}$ and, owing to (3.15), $f(v_h, \delta_h v_h, t, x)$ satisfies

$$|f| \leq N_1(\bar{H} + K + |v_h| + M_h), \tag{4.12}$$

where $N_1 = N(d)K_0'$. This properly of f implies that there exist functions b_k, $k = 1,...,d$, c, and θ with values in $[-N_1, N_1]$ such that

$$f(v_h, \delta_h v_h, t, x) = cv_h + \sum_{k=1}^d b_k \delta_{h,e_k}v_h + \theta(\bar{H} + K),$$

so that $w_h(t,x) := v_h(t,x)\exp(N_1 t)$ satisfies

$$\partial_t w_h + \sum_{k=1}^m a_k \Delta_{h,l_k}w_h + \sum_{k=1}^d b_k \delta_{h,l_k}w_h + (c - N_1)w_h + \theta(\bar{H} + K)e^{N_1 t} = 0.$$

After that (4.9) is proved by using the barrier function Φ from Lemma 8.8 of [8] and the comparison principle (see, for instance, Section 5 of [2]).

Having in mind translations and the continuity of $\partial_t v_h$ with respect to t we see that it suffices to prove (4.10) on $(0,T) \times (\bar\Omega \cap \Lambda_\infty^h)$. Introduce

$$Q^o = \{(0,T) \times [\Omega^h \cap \Lambda_\infty^h]\} \cap G.$$

Since v_h satisfies (4.6), estimate (4.10) obviously holds on

$$(0,T) \times (\bar\Omega \setminus \Omega^h).$$

On $(0,T) \times [\Omega^h \cap \Lambda_\infty^h] \setminus Q^o$, we have

$$(\hat\delta/2) \sum_k |\Delta_{h,l_k} v_h| \leq \bar H + K + K_0(|v_h(t,x)| + M_h) \qquad (4.13)$$

which together with (4.9), (4.11), and (4.12) implies that (4.10) holds on $(0,T) \times [\bar\Omega \cap \Lambda_\infty^h] \setminus Q^o$. Therefore, it remains to establish (4.10) on Q^o assuming that $Q^o \neq \emptyset$.

Recall that (4.8) holds. Furthermore, every x-section of Q^o is the union of open intervals on which $\partial_t v_h$ is Lipschitz continuous by virtue of (4.8). By subtracting the left-hand sides of (4.8) evaluated at points t and $t + \varepsilon$, then transforming the difference by using Hadamard's formula, and finally dividing by ε and letting $\varepsilon \to 0$, we get that there exist functions a_k such that $\hat\delta/2 \leq a_k \leq 2\hat\delta^{-1}$ and on every x-section of Q^o (a.e.) we have

$$\partial_t(\partial_t v_h) + a_k \Delta_{h,l_k}(\partial_t v_h) = 0.$$

By Lemma 4.2 of [2] this yields

$$\sup_{Q^o} |\partial_t v_h| \leq \sup_{(0,T] \times [\Omega \cap \Lambda_\infty^h] \setminus Q^o} |\partial_t v_h|,$$

which implies (4.10) on Q^o. The lemma is proved.

Lemma 4.4. *There are constants $h_0 > 0$ and N such that for all $h \in (0, h_0]$ and $r = 1, ..., m$*

$$(\rho - 6\lambda h)|\Delta_{h,l_r} v_h| \leq N(\bar M_h + \bar H + K + \|g\|_{C^{1,2}(\bar\Omega_T)}) \qquad (4.14)$$

on $[0,T] \times \Omega^h$ (remember that λ is the radius of B).

Proof. As in the proof of Lemma 4.3 we will focus on proving (4.14) in $(0,T) \times [\Omega^h \cap \Lambda_\infty^h]$. Then fix r and define

$$Q^o := \{(0,T) \times [\Omega^{3h} \cap \Lambda_\infty^h]\} \cap G.$$

Obviously, $Q \subset [0,T] \times \Omega^h$. Next, if $t \in (0,T)$, and $x \in \Omega^h \cap \Lambda_\infty^h$ is such that $(t,x) \notin Q^o$, then either $x \notin \Omega^{3h}$, so that $\rho(x) \leq 3\lambda h$ and (4.14) holds, or else $x \in \Omega^{3h}$ but (4.13) is valid, in which case (4.14) holds again.

Thus we need only prove (4.14) on Q^o assuming, of course, that $Q^o \neq \emptyset$. We know that (4.8) holds and the left-hand side of (4.8) is nonpositive in $Q \setminus Q^o$.

To proceed further observe a standard fact that there are constants $\mu_0 \in (0, 1]$ and $N \in [0, \infty)$ depending only on Ω such that for any $\mu \in (0, \mu_0]$ there exists an $\eta_\mu \in C_0^\infty(\Omega)$ satisfying

$$\eta_\mu = 1 \quad \text{on} \quad \Omega^{2\mu}, \quad \eta_\mu = 0 \quad \text{outside} \quad \Omega^\mu,$$

$$|\eta_\mu| \le 1, \quad |D\eta_\mu| \le N/\mu, \quad |D^2\eta_\mu| \le N/\mu^2. \tag{4.15}$$

By Lemma 4.2 on $Q^o \cap \Omega_T^{2\mu}$

$$[(\Delta_{h,l_r} v_h)^-]^2 \le \sup_{Q \setminus Q^o} \eta_\mu [(\Delta_{h,l_r} v_h)^-]^2 + N\mu^{-2} \bar{M}_h^2.$$

While estimating the last supremum we will only concentrate on $h_0 \le \mu_0/3$ and $\mu \in [3h, \mu_0]$, when $\eta_\mu = 0$ outside Ω^{3h}. In that case, for any $(s, y) \in Q \setminus Q^o$, either $y \notin \Omega^{3h}$ implying that

$$\eta_\mu[(\Delta_{h,l_r} v_h)^-]^2(s, y) = 0,$$

or $y \in \Omega^{3h} \cap \Lambda_\infty^h$ but (4.13) holds at (s, y), or else $((s, y) \notin Q^o$ and) there is a sequence $s_n \uparrow s$ such that $(s_n, y) \in Q^o$.

The third possibility splits into two cases: 1) $s = T$, 2) $s < T$. In case 1 we have

$$|\Delta_{h,l_r} v_h(s, y)| = |\Delta_{h,l_r} g(s, y)| \le N\|g\|_{C^{1,2}(\bar{\Omega}_T)}.$$

In case 2, owing to the continuity of $\Delta_{h,l_r} v_h(s, y)$ with respect to s, estimate (4.13) holds again at (s, y).

It follows that as long as $h \in (0, h_0]$, $(t, x) \in Q^o \cap \Omega_T^{2\mu}$, and $\mu \in [3h, \mu_0]$ we have

$$(\Delta_{h,l_r} v_h)^-(t, x) \le N\mu^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)} + \bar{M}_h). \tag{4.16}$$

If $(t, x) \in Q^o$ and x is such that $\rho(x) \ge 6\lambda h$, take $\mu = \mu_0 \wedge (\rho(x)/(2\lambda))$, which is bigger than $3h$ provided that $h \le \mu_0/3$. In that case also $\rho(x) \ge 2\lambda\mu$, so that $x \in \Omega^{2\mu}$ and we conclude from (4.16) that

$$\rho(x)(\Delta_{h,l_r} v_h)^-(t, x) \le N(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)} + \bar{M}_h),$$

$$(\rho(x) - 6\lambda h)(\Delta_{h,l_r} v_h)^-(t, x) \le N(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)} + \bar{M}_h)$$

for $(t, x) \in Q^o$ such that $\rho(x) \ge 6\lambda h$. However, the second relation here is obvious for $\rho(x) \le 6\lambda h$.

As a result of all the above arguments we see that

$$(\rho - 6\lambda h)(\Delta_{h,l_r} v_h)^- \le N(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)} + \bar{M}_h) \tag{4.17}$$

holds in $(0, T) \times [\Omega^h \cap \Lambda_\infty^h]$ for any r whenever $h \in (0, h_0]$.

Finally, since $\partial_t v_h + P_h[v_h] \le K$ in $(0, T) \times \Omega^h$, we have that

$$2\hat{\delta}^{-1} \sum_r (\Delta_r v_h)^+ \le -\partial_t v_h + (\hat{\delta}/2) \sum_r (\Delta_r v_h)^- + K,$$

which after being multiplied by $\rho - 6h$ along with (4.17) and (4.10) leads to (4.14) on $(0, T) \times [\Omega^h \cap \Lambda_\infty^h]$. Thus, as is explained at the beginning of the proof, the lemma is proved.

Our final estimates hinge on the first-order difference estimates.

Lemma 4.5. *There is a constant N such that for all sufficiently small $h > 0$ the estimates*

$$|v_h|, |\partial_t v_h|, |\delta_{h,l_k} v_h|, (\rho - 6\lambda h)|\Delta_{h,l_k} v_h| \le N(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}) \quad (4.18)$$

hold in $[0, T] \times \Omega^h$ for all k.

Proof. The first estimate in (4.18) is obtained in Lemma 4.3. Owing to Lemmas 4.3 and 4.4, the remaining estimates would follow if we can prove that

$$|\delta_{h,l_k} v_h| \le N(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}) \quad (4.19)$$

in $[0, T] \times \Omega^h$ for all k.

We are going to use interpolation inequalities. Note that if we have a function $u(i)$ on a set $-r + 1, ..., 0, 1, ..., r$, where $r \ge 2$ is an integer, which satisfies

$$u(i + 1) - 2u(i) + u(i - 1) \ge -N_1 \quad (4.20)$$

for $i = -r + 2, ..., r - 1$, where N_1 is a constant, then

$$u(i + 1) - u(i) \ge u(i) - u(i - 1) - N_1.$$

It follows that $w(i) := u(i + 1) - u(i) + N_1 i$ is an increasing function of $i = -r + 1, ..., r - 1$. In particular,

$$u(1) - u(0) = w(0) \le \frac{1}{r - 1} \sum_{i=1}^{r-1} w(i)$$

$$= \frac{1}{r - 1} \sum_{i=1}^{r-1} (u(i + 1) - u(i) + N_1 i) = \frac{1}{r - 1}(u(r) - u(1)) + \frac{1}{2} N_1 r.$$

On the other hand,

$$u(1) - u(0) \ge \frac{1}{r - 1} \sum_{i=-r+1}^{-1} (u(i + 1) - u(i) + N_1 i)$$

$$= \frac{1}{r - 1}(u(0) - u(-r + 1)) - \frac{1}{2} N_1 r.$$

It follows that

$$|u(1) - u(0)| \le \frac{1}{2} N_1 r + \frac{2}{r - 1} \max\{|u(i)| : i = -r + 1, ..., r\},$$

and for any function w (use that $(r - 1)^{-1} \le 2r^{-1}$ for $r \ge 2$)

$$|w(1) - w(0)| \le \frac{r}{2} \max_{|i| \le r} |w(i + 1) - 2w(i) + w(i - 1)| + \frac{4}{r} \max_{|i| \le r} |w(i)|. \quad (4.21)$$

Now fix an $\varepsilon \in (0, 1]$ and set

$$n(\varepsilon) = 10/\varepsilon.$$

Observe that if $x \in \Omega^{n(\varepsilon)h}$ and we take $r = [(\varepsilon\rho(x) - 6\lambda h)(2\lambda h)^{-1}]$ ($[a]$ is the integer part of a), then $r \geq 2$ and

$$\varepsilon[\rho(x + ihl_k) - 6\lambda h] \geq r\lambda h \tag{4.22}$$

for $|i| \leq r$ since $\rho(x + ihl_k) \geq \rho(x) - \lambda rh$ and

$$\varepsilon\rho(x) - (1 + \varepsilon)r\lambda h \geq \varepsilon\rho(x) - 2r\lambda h \geq 6\lambda h.$$

In particular, $x + ihl_k \in \Omega^h$ for $|i| \leq r$ and it makes sense applying (4.21) to $w(i) = v_h(t, x + ihl_k) - g(t, x + ihl_k)$ with $t \in (0, T)$, which yields

$$|\delta_{h,l_k}(v_h - g)(t, x)| \leq \frac{1}{2}rh \max_{|i| \leq r} |\Delta_{h,l_k}(v_h - g)(t, x + ihl_k)|$$

$$+ \frac{4}{rh} \max_{|i| \leq r} |(v_h - g)(t, x + ihl_k)|. \tag{4.23}$$

Also notice that for $x \in \Omega^{n(\varepsilon)h}$

$$2r\lambda h \geq \varepsilon\rho(x) - 8\lambda h, \quad 10\lambda h < \varepsilon\rho(x), \quad 10r\lambda h \geq \varepsilon\rho(x),$$

$$\rho(x + ihl_k) \leq \rho(x) + r\lambda h \leq rh(10\lambda\varepsilon^{-1} + \lambda) \leq rh11\lambda\varepsilon^{-1}. \tag{4.24}$$

With so specified r it follows from (4.22), (4.24), (4.23) and Lemmas 4.3 and 4.4 that for all sufficiently small h and $x \in \Omega^{n(\varepsilon)h}$

$$|\delta_{h,l_k}(v_h - g)(t, x)| \leq N\varepsilon(\bar{M}_h + \bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)})$$

$$+ N\varepsilon^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}).$$

Hence, for all sufficiently small h we have

$$\bar{M}_h = \sup_{[0,T] \times \Omega^h} \sum_{k=1}^{m} |\delta_{h,l_k}v_h| \leq N_1\varepsilon(\bar{M}_h + \bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)})$$

$$+ N\varepsilon^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}) + \sup_{[0,T] \times (\Omega^h \setminus \Omega^{n(\varepsilon)h})} \sum_{k=1}^{m} |\delta_{h,l_k}(v_h - g)|,$$

where the last term is dominated by

$$Nn(\varepsilon)(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)})$$

in light of (4.9). To finish proving (4.19) it now remains only pick and fix $\varepsilon \in (0, 1]$ so that $N_1\varepsilon \leq 1/2$. The lemma is proved.

Mimicking the proof of Corollary 2.7 of [11], we obtain the following corollary of (4.18). Note that here assumptions (3.1) and (3.2) play a crucial role and in (3.1) only the Lipschitz continuity in x is needed.

Corollary 4.6. *There are constants $h_0 > 0$ and M, depending only on Ω, Λ, d, K_0, T, δ, and N', such that for all $h \in (0, h_0]$, $t \in [0, T]$, and $x, y \in \Omega$, we have*

$$|v_h(t, x) - v_h(t, y)| \leq M(|x - y| + h).$$

Proof of Theorem 3.3. In what concerns the first assertion of Theorem 2.1 and estimates (2.2) one derives them in the same way as Theorem 5.2 in [2] is proved (and using Lemma 4.2 [2] with c of any sign).

To prove (2.3) observe that

$$\max(H(v(t,x), Dv(t,x), u'', t, x), P(u'') - K) = P(u'') + G(u'', t, x),$$

where

$$G(u'', t, x) = (H(v(t,x), Dv(t,x), u'', t, x) - P(u'') + K)^+ - K.$$

Furthermore, in light of (2.2) and (3.17)

$$|G(u'', t, x)| \le (H(v(t,x), Dv(t,x), u'', t, x) - P(u'') + K)^+ + K$$

$$\le \bar{H} + K_0\big(|v(t,x)| + |Dv(t,x)|\big) + 2K \le N, \qquad (4.25)$$

where N is a constant like the right-hand side of (2.2). Then set

$$G(t,x) = G(D^2 v(t,x), t, x)$$

and observe that our function v satisfies the equation

$$\partial_t u(t,x) + P(D^2 u(t,x)) + G(t,x) = 0 \qquad (4.26)$$

Since P is convex with respect to u'' and $G(t,x)$ is bounded, due to Theorem 1.2 of [3] there is a unique solution $u \in W_p^{1,2}(\Omega_T)$ of (4.26) with boundary condition $u = g$ on $\partial' \Omega_T$. By uniqueness of $W_{d+1,\text{loc}}^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$-solutions we obtain $u = v \in W_p^{1,2}(\Omega_T)$. This allows us to apply a priori estimates from Theorem 1.2 of [3] and along with (4.25) proves (2.3).

Finally, to prove (2.4) introduce

$$F_K = \max(H, P - K)$$

and notice that since $|F_K(u', 0, t, x)| \le \bar{H} + K_0|u'|$, there exist functions $b_1, ..., b_d, c,$ and f such that

$$|b_i|, |c| \le K_0, \quad |f| \le \bar{H},$$

$$F_K(v(t,x), Dv(t,x), 0, t, x) = b_i(t,x) D_i v(t,x) - c(t,x) v(t,x) + f(t,x),$$

so that

$$0 = \partial_t v(t,x) + F_K[v](t,x) - F_K(v(t,x), Dv(t,x), 0, t, x)$$

$$+ b_i(t,x) D_i v(t,x) - c(t,x) v(t,x) + f(t,x)$$

$$= \partial_t v + a_{ij} D_{ij} v + b_i(t,x) D_i v(t,x) - c(t,x) v(t,x) + f(t,x),$$

where (a_{ij}) is a $\mathbb{S}_{\hat{\delta}/2}$-valued function. Now (2.4) follows from classical results (see, for instance, [6], [12]). The theorem is proved.

5. Proof of Theorem 2.2

As in Section 3 we easily reduce proving Theorem 2.2 to proving the following.

Theorem 5.1. *Suppose that $g \in C^2(\mathbb{R}^d)$ and Assumption 2.1 is satisfied with $\delta/2$ in place of δ. Also assume that (3.15) (and hence (3.16)) holds at all points of differentiability of $H(u, t, x)$ with respect to u. Finally, assume that estimates (3.1) and (3.2) with a constant N' and (3.17) hold for any $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, and u, v. Then the assertions of Theorem 2.2 hold true.*

To prove Theorem 5.1 consider the equation

$$\partial_t v + H_{K,h}[v] = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^d \tag{5.1}$$

with terminal condition

$$v(T, x) = g(x) \quad \text{on} \quad \mathbb{R}^d \tag{5.2}$$

In view of Picard's method of successive approximations and the Lipschitz continuity of $\mathcal{H}$ with respect to u uniform with respect to (t, x), for any $h > 0$ there exists a unique bounded solution $v = v_h$ of (5.1)–(5.2). Furthermore, $\partial_t v_h$ is bounded and continuous.

We need a version of Lemma 4.2 of [2] for unbounded domains, in which $Q^o, \hat{Q}^o, Q$ are generic objects described in Section 4 before assumption (4.7) was made.

Lemma 5.2. *Let $(a, b, c)(t, x)$ be a bounded $\mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}$-valued function on $\mathbb{R}^{d+1}$ satisfying $a_k \geq 0$ and $hb_k^- \leq a_k$ and let $v(t, x)$ be a bounded function in Q which is absolutely continuous with respect to t on each open interval belonging to $Q^o_{|x}$ and for any $x \in \Lambda^h_\infty$ satisfies*

$$\partial_t v + Lv := \partial_t v + \sum_{k=1}^m a_k \Delta_{h,l_k} v + \sum_{k=1}^d b_k \delta_{h,l_k} v - cv = -\eta$$

(a.e.) on each $Q^o_{|x}$, where $\eta = \eta(t, x)$ is a bounded function. Redefine v if necessary for $(t, x) \in \hat{Q}^o \setminus Q^o$ so that

$$v(t, x) = \varlimsup_{s \uparrow t, (s,x) \in Q^o} v(s, x).$$

Then for h sufficiently small in Q^o we have

$$v \leq T e^{\bar{c}T} \sup_{Q^o} \eta_+ + e^{\bar{c}T} \sup_{Q \setminus Q^o} v^+,$$

where $\bar{c} = \sup c^-$,

Proof. First, as in [2] we reduce the general case to the one where $c \geq 0$. Then, by considering

$$v(t, x) - (T - t) \sup_{Q^o} \eta^+ - \sup_{Q \setminus Q^o} v^+,$$

we reduce the general case to the one with $\eta \leq 0$ and $v \leq 0$ on $Q \setminus Q^o$.

Observe that for $\zeta(x) = \cosh|x|$ we have

$$|D\zeta| + |D^2\zeta| \le N'\zeta,$$

where N' depends only on d. It follows that for a different N', $h \in (0,1)$, and $k = 1, ..., m$

$$|\delta_{h,l_k}\zeta| + |\Delta_{h,l_k}\zeta| \le N'\zeta.$$

Hence, the bounded function $w := v\zeta^{-1}$ satisfies

$$-\eta = \partial_t(w\zeta) + L(w\zeta) = \zeta\partial_t w + \zeta\sum_{k=1}^{m} a_k\Delta_{h,l_k}w$$

$$+\zeta\sum_{k=1}^{m} a_k[c_{-k}\delta_{h,-l_k}w + c_k\delta_{h,l_k}w] + \zeta\sum_{k=1}^{d} \bar{b}_k\delta_{h,l_k}w + \zeta\bar{c}w$$

where $c_{\pm k} = \zeta^{-1}\delta_{h,\pm l_k}\zeta$, $\bar{b}_k = b_k\zeta^{-1}T_{h,l_k}\zeta$,

$$\bar{c} = -c + \zeta^{-1}\sum_{k=1}^{m}\Delta_{h,l_k}\zeta + \zeta^{-1}\sum_{k=1}^{d} b_k\delta_{h,l_k}\zeta.$$

It follows that for any constant $\lambda > 0$ we have

$$\partial_t(we^{\lambda(T-t)}) + \sum_{k=1}^{m} a_k\Delta_k(we^{\lambda(T-t)}) + \sum_{k=1}^{d} \bar{b}_k\delta_{h,l_k}(we^{\lambda(T-t)})$$

$$+\sum_{k=1}^{m} a_k[c_{-k}\delta_{h,-l_k}(we^{\lambda(T-t)}) + c_k\delta_{h,l_k}w] + (\bar{c} - \lambda)(we^{\lambda(T-t)}) \ge 0. \quad (5.3)$$

For λ sufficiently large and h sufficiently small we have $\bar{c} - \lambda \le 0$ and the coefficients in (5.3) satisfy other conditions of Lemma 4.2 of [2] which allows us to conclude that for any $R \in (0, \infty)$ on $Q^o \cap [(0,T) \times B_R]$ we have

$$w(t,x)e^{\lambda(T-t)} \le \sup\{w^+(s,x)e^{\lambda(T-s)} : (s,x) \in Q, |x| \ge R\}.$$

Here the right-hand side goes to zero as $R \to \infty$ since $|w| = |v|\zeta^{-1}$ and v is bounded. Hence $w \le 0$ and this proves the lemma.

Corollary 5.3. *There exists a constant N depending only on d and K_0 such that for all sufficiently small h we have*

$$|v_h| \le Ne^{NT}(\bar{H} + K + \sup|g|). \quad (5.4)$$

This corollary is obtained from Lemma 5.2 by repeating the first part of the proof of Lemma 4.3.

Lemma 5.4. *There exists a constant N depending only on d, δ, T, and K_0 such that for all sufficiently small h we have*

$$|\partial_t v_h| \le N(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)} + \sup_{(0,T)\times\mathbb{R}^d}\sum_{k=1}^{m}|\delta_{h,l_k}v_h|), \quad (5.5)$$

$$\sum_{k=1}^{m} |\Delta_{h,l_k} v_h| \leq N(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)} + \sup_{(0,T) \times \mathbb{R}^d} \sum_{k=1}^{m} |\delta_{h,l_k} v_h|) \qquad (5.6)$$

on $(0,T) \times \mathbb{R}^d$.

Proof. One proves (5.5) in the same way as (4.10) with the only difference that instead of Lemma 4.2 of [2] one uses Lemma 5.2.

In case of (5.6) we add to (4.8) the fact that the left-hand side of (4.8) is nonpositive outside Q^o. Hence, for any $r \in \{1, ..., m\}$ on Q^o there exist functions a_k satisfying $\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1}$ such that on every x-section of Q^o (a.e.) we have

$$\partial_t (\Delta_{h,l_r} v_h) + a_k \Delta_{h,l_k} (\Delta_{h,l_r} v_h) \leq 0.$$

It follows by Lemma 5.2 that in Q^o

$$(\Delta_{h,l_r} v_h)^- \leq \sup_{(0,T] \times \Lambda_\infty^h \setminus Q^o} (\Delta_{h,l_r} v_h)^-.$$

The continuity of $\Delta_{h,l_r} v_h$ with respect to t and the definition of Q^o show that $(\Delta_{h,l_r} v_h)^-$ is dominated by the right-hand side of (5.6). Then equation (4.11) combined with estimates (4.12), (5.5), and (5.4) allow us to conclude that also $(\Delta_{h,l_r} v_h)^+$ is dominated by the right-hand side of (5.6). This proves the lemma.

Our next step is to exclude $|\delta_{h,l_k} v_h|$ from the right-hand side of (5.5) and (5.6) by using interpolation, that is by using (4.21), which for $w(i) = v_h(t, x + ihl_k)$, where $(t,x) \in (0,T) \times \mathbb{R}^d$, $h < 1$, and integer $r \geq 2$ yields that

$$|\delta_{h,l_k} v_h(t,x)| \leq \frac{1}{2} rh \max_{|i| \leq r} |\Delta_{h,l_k} v_h(t, x + ihl_k)| + \frac{4}{rh} \max_{|i| \leq r} |v_h(t, x + ihl_k)|.$$

In light of the arbitrariness of $r \geq 2$ and (5.4) and (5.6) we conclude that for any $\varepsilon \geq 2h$

$$|\delta_{h,l_k} v_h| \leq N\varepsilon^{-1}(\bar{H} + K + \sup |g|)$$

$$+ N\varepsilon(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)} + \sup_{(0,T) \times \mathbb{R}^d} \sum_{k=1}^{m} |\delta_{h,l_k} v_h|).$$

It follows that for all sufficiently small h we have

$$\sup_{(0,T) \times \mathbb{R}^d} \sum_{k=1}^{m} |\delta_{h,l_k} v_h| \leq N(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)}), \qquad (5.7)$$

$$\sup_{(0,T) \times \mathbb{R}^d} \left(|v_h| + |\partial_t v_h| + \sum_{k=1}^{m} |\Delta_{h,l_k} v_h| \right) \leq N(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)}). \qquad (5.8)$$

Mimicking the proof of Corollary 2.7 of [11], we obtain the following corollary from (5.7) and (5.8).

Corollary 5.5. *There is a constant M, which may depend on N', such that for all sufficiently small h, $t \in (0,T]$, and $x, y \in \Omega$, we have*

$$|v_h(t,x) - v_h(t,y)| \leq M(|x-y| + h).$$

After that one finishes the proof of Theorem 2.2 in the same way as Theorem 3.3 is proved.

6. Proof of Theorem 2.3

By the maximum principle v_K decreases as K increases. Estimate (2.4) guarantees that v_K converges uniformly to a function $v \in C(\bar{\Omega}_T)$. To prove that v is an L_{d+1}-viscosity solution we need the following, in which

$$C_r = (0, r^2) \times B_r, \quad C_r(t, x) = (t, x) + C_r.$$

Lemma 6.1. *There is a constant N depending only on d, δ, and the Lipschitz constant of H with respect to $(u'_1, ..., u'_d)$ such that for any $r \in (0, 1]$ and $C_r(t, x)$ satisfying $C_r(t, x) \subset \Omega_T$ and $\phi \in W^{1,2}_{d+1}(C_r(t, x))$ we have on $C_r(t, x)$ that*

$$v \le \phi + Nr^{d/(d+1)} \|(\partial_t \phi + H[\phi])^+\|_{L_{d+1}(C_r(t,x))} + \max_{\partial' C_r(t,x)} (v - \phi)^+. \quad (6.1)$$

$$v \ge \phi - Nr^{d/(d+1)} \|(\partial_t \phi + H[\phi])^-\|_{L_{d+1}(C_r(t,x))} - \max_{\partial' C_r(t,x)} (v - \phi)^-. \quad (6.2)$$

Proof. Observe that

$$-\partial_t \phi - \max(H[\phi], P[\phi] - K) = -\partial_t \phi - \max(H[\phi], P[\phi] - K)$$

$$+\partial_t v_K + \max(H[v_K], P[v_K] - K)$$

$$= \partial_t(v_K - \phi) + a_{ij} D_{ij}(v_K - \phi) + b_i D_i(v_K - \phi) - c(v_K - \phi),$$

where $a = (a_{ij})$ is a $d \times d$ symmetric matrix-valued function whose eigenvalues are in $[\hat{\delta}/2, 2\hat{\delta}^{-1}]$, b_i are bounded functions, and $c \ge 0$. It follows by Lemma 2.1 and Remark 1.1 of [7] with

$$u = v_K - \phi - \max_{\partial' C_r(t,x)} (v_K - \phi)^+$$

that for $r \in (0, 1]$

$$v_K \le \phi + \max_{\partial' C_r(t,x)} (v_K - \phi)^+$$

$$+ Nr^{d/(d+1)} \|(\partial_t \phi + \max(H[\phi], P[\phi] - K))^+\|_{L_{d+1}(C_r(t,x))}, \quad (6.3)$$

where the constant N is of the type described in the statement of the present lemma. We obtain (6.1) from (6.3) by letting $K \to \infty$. In the same way (6.2) is established. The lemma is proved.

Now we can prove that v is an L_{d+1}-viscosity solution. Let $(t_0, x_0) \in \Omega_T$ and $\phi \in W^{1,2}_{d+1,loc}(\Omega_T)$ be such that $v - \phi$ attains a local maximum at (t_0, x_0) and $v(t_0, x_0) = \phi(t_0, x_0)$. Then for $\varepsilon > 0$ and all small $r > 0$ for

$$\phi_{\varepsilon,r}(t, x) = \phi(t, x) + \varepsilon(|x - x_0|^2 + t - t_0 - r^2)$$

we have that

$$\max_{\partial' C_r(t_0,x_0)} (v - \phi_{\varepsilon,r})^+ = 0.$$

Hence, by Lemma 6.1

$$\varepsilon r^2 = (v - \phi_{\varepsilon,r})(t_0, x_0) \leq N r^{d/(d+1)} \|(\partial_t \phi_{\varepsilon,r} + H[\phi_{\varepsilon,r}])^+\|_{L_{d+1}(C_r(t_0,x_0))},$$

$$N r^{-(d+2)} \|(\partial_t \phi_{\varepsilon,r} + H[\phi_{\varepsilon,r}])^+\|_{L_{d+1}(C_r(t_0,x_0))}^{d+1} \geq \varepsilon^{d+1}.$$

By letting $r \downarrow 0$ and using the continuity of $H(u, t, x)$ in u'_0, which is assumed to be uniform with respect to other variables, we obtain

$$N \lim_{r \downarrow 0} \operatorname*{ess\,sup}_{C_r(t_0,x_0)} (\partial_t \phi_\varepsilon + H[\phi_\varepsilon]) \geq \varepsilon.$$

where $\phi_\varepsilon = \phi + \varepsilon(|x - x_0|^2 + t - t_0)$. After that letting $\varepsilon \downarrow 0$ proves that v is an L_{d+1}-viscosity subsolution. The fact that it is also an L_{d+1}-viscosity supersolution is proved similarly on the basis of (6.2).

Finally, we prove that v is the maximal continuous L_{d+1}-viscosity subsolution. Let u be an L_{d+1}-viscosity subsolution of (1.1) of class $C(\bar{\Omega}_T)$. Then, as is easy to see, for any $K \geq 0$, $u - v_K$ is an L_{d+1}-viscosity subsolution of

$$\partial_t w + F[w] = -h_K,$$

where $F[w] := H[w + v_K] - H[v_K]$, so that $F[0] = 0$, and $h_K(t, x) = H[v_K]$. Since $h_K \leq 0$, we conclude by Proposition 2.6 of [1] that, if, additionally, $u = g$ on $\partial \Omega_T$, then $u - v_K \leq 0$ in Ω_T. Now it only remains to let $K \to \infty$. The theorem is proved.

Remark 6.1. As follows from [1] continuous L_{d+1}-viscosity subsolutions u of (1.1) satisfy (6.1) with u in place of v for any $\phi \in W_{d+1}^{1,2}(C_r(t, x))$ whenever $r \in (0, 1]$ and $C_r(t, x) \subset \Omega_T$. Therefore, this relation can be taken as an equivalent definition of what L_{d+1}-viscosity subsolutions are. A nice feature of (1.1) is that it is satisfied for any $\phi \in W_{d+1}^{1,2}(C_r(t, x))$ iff it is satisfied for any $\phi \in C^{1,2}(\bar{C}_r(t, x))$.

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